Final Exam (B5709514, Spring 2014)

June 19, 2014. **4:30 pm – 5:45 am**

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Provide full details of your solutions

1. Solve the following third-order ODE having initial values. (*10 points*)

$$y''' - 4y = 10\cos x + 5\sin x$$

$$y(0) = 3,\ y'(0) = -2,\ y''(0) = -1$$

i) y_h: $y''' - 4y = 0$

characteristic eq: $\lambda^3 - 4\lambda = 0 \Rightarrow \lambda = 0, \pm 2$

$\therefore y_h = C_1 + C_2 e^{2x} + C_3 e^{-2x}$

3

ii) y_p: 'method of undetermined coefficient'

$y_p = A\cos x + B\sin x \Rightarrow$ original ODE

$A = 1,\ B = -2$

$\therefore y_p = \cos x - 2\sin x$

6

$y = y_h + y_p$

$= C_1 + C_2 e^{2x} + C_3 e^{-2x} + \cos x - 2\sin x$ 7

we apply $y(0) = 3$, $y'(0) = -2$, $y''(0) = -1$

to obtain

$C_1 = 2,\ C_2 = 0,\ C_3 = 0$

$\therefore y = 2 + \cos x - 2\sin x$

10

2. Obtain the general solutions of this system of ODEs. (*10 points*)

$$y_1' = 4y_1 + y_2$$

$$y_2' = -y_1 + 2y_2$$

$\Rightarrow \underset{\sim}{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underset{\sim}{y}$

let $\underset{\sim}{y} = \underset{\sim}{x} e^{\lambda t}$

$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \det \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda = 3$ (double root)

2

The corresponding eigenvalue equation is

$(\underset{\sim}{A} - 3\underset{\sim}{I}) \cdot \underset{\sim}{x} = \underset{\sim}{0}$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{\sim}{0}$

$\therefore \underset{\sim}{x}_{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underset{\sim}{y}_{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

4

$\underset{\sim}{y}_{(2)} = \underset{\sim}{x}_{(1)} t \cdot e^{\lambda_1 t} + \underset{\sim}{u} e^{\lambda_1 t}$, $\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\underset{\sim}{y}_{(2)}' = \underset{\sim}{x}_{(1)} \cancel{e^{\lambda_1 t}} + \lambda_1 \underset{\sim}{x}_{(1)} t \cancel{e^{\lambda_1 t}} + \lambda_1 \underset{\sim}{u} e^{\lambda_1 t}$

$\Leftrightarrow \underset{\sim}{A}\, \underset{\sim}{y}_{(2)} = \underset{\sim}{A}\underset{\sim}{x}_{(1)} t e^{\lambda_1 t} + \underset{\sim}{A}\underset{\sim}{u} e^{\lambda_1 t}$

We thus obtain $\underset{\sim}{x} + \lambda_{(1)} \underset{\sim}{u} = \underset{\sim}{A}\underset{\sim}{u}$

or $\underset{\sim}{x} = (\underset{\sim}{A} - \lambda_{(1)} \underset{\sim}{I}) \cdot \underset{\sim}{u}$

$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$1 = u_1 + u_2$

Any combination that satisfies $1 = u_1 + u_2$ can be $\underset{\sim}{u}$, but we take a simple one.

$\underset{\sim}{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

7

$\therefore \underset{\sim}{y}_{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$

9

$\therefore \underset{\sim}{y} = C_1 \underset{\sim}{y}_{(1)} + C_2 \underset{\sim}{y}_{(2)}$

$= C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + C_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$

10

3. Determine the type (*3 points*) and stability (*2 points*) of the critical point for following systems of ODEs. (*5 points per each*)

1) $y_1' = -2y_1 - 6y_2$
$y_2' = -8y_1 - 4y_2$

2) $y_1' = -4y_2$
$y_2' = \sin y_1$

1) $\underset{\sim}{y}' = \begin{bmatrix} -2 & -6 \\ -8 & -4 \end{bmatrix} \underset{\sim}{y}$

$p = \underset{<0}{-6}, \quad q = \underset{<0}{-40}, \quad \Delta = p^2 - 4q = 196 > 0$

saddle point, unstable

~~no partial points~~

2) critical points: $y_1' = 0, \; y_2' = 0$ 1

$(y_1, y_2) = (0, n\pi) \quad (n = 0, \pm1, \pm2, \cdots)$

i) at $(y_1, y_2) = (0, 2m\pi)$

$\sin y_1 \sim y_1$

then $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \underset{\sim}{y} \Rightarrow p = 0, \; q > 0, \; \Delta < 0$

stable center 3

ii) at $(y_1, y_2) = (0, (2m+1)\pi)$

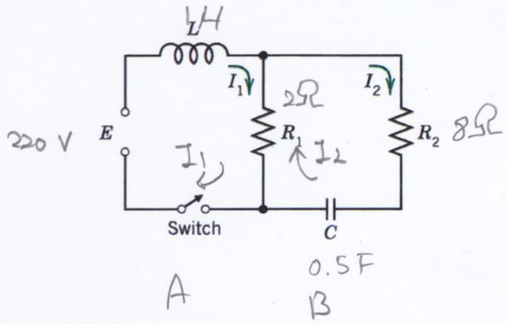
$y_1 = \theta - \pi$, in this case $\sin\theta = -\sin y_1 \sim -y_1$

then $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix} \underset{\sim}{y} \Rightarrow p = 0, \; q < 0, \; \Delta > 0$

unstable saddle points

5

4. We have a RLC circuit with $R_1 = 2\ \Omega$, $R_2 = 8\ \Omega$, $L = 1$ H, and $C = 0.5$ F, E = 220 V. Answer following questions. (*20 points*)



1) Derive the governing equations (two ODEs) for I_1 and I_2. (*8 points*)

for circuit A,

$220 - I_1' - 2(I_1 - I_2) = 0$ 2

for circuit B,

$-8I_2 - 2\int I_2\, dt - 2(I_2 - I_1) = 0$ 4

$\therefore \begin{cases} I_1' + 2I_1 - 2I_2 = 220 \Rightarrow I_1' = -2I_1 + 2I_2 + 220 \\ 2I_1' - 10I_2' - 2I_2 = 0 \Rightarrow I_2' = -0.4I_1 + 0.2I_2 + \cancel{48}\ 44 \end{cases}$

2) Find the solutions of I_1 and I_2. (*12 points*)

$\underset{\sim}{I}' = \begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix}}_{\equiv \underset{\sim}{A}} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 220 \\ 44 \end{bmatrix}$

i) $\underset{\sim}{I}_h$:

$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$

$\lambda = -0.9 \pm \sqrt{0.41}$

corresponding eigen value equation

$(\underset{\sim}{A} - \lambda \underset{\sim}{I}) \cdot \underset{\sim}{X} = 0$

$\underset{\sim}{X} = \begin{bmatrix} 2 \\ 1.1 + \sqrt{0.41} \end{bmatrix} \; \& \; \begin{bmatrix} 2 \\ 1.1 - \sqrt{0.41} \end{bmatrix}$

$\therefore \underset{\sim}{I}_h = C_1 \begin{bmatrix} 2 \\ 1.1 + \sqrt{0.41} \end{bmatrix} e^{(-0.9 + \sqrt{0.41})t} + C_2 \begin{bmatrix} 2 \\ 1.1 - \sqrt{0.41} \end{bmatrix} e^{(-0.9 - \sqrt{0.41})t}$

or (-0.26) (or -1.54)

5

ii) $\underset{\sim}{I}_p = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow$ insert to the above ODE.

then $a = 100$, $b = 0$ $\therefore \underset{\sim}{I}_p = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ 8

Thus. $\underset{\sim}{I} = \underset{\sim}{I}_h + \underset{\sim}{I}_p$

$= C_1 \begin{bmatrix} 2 \\ 1.74 \end{bmatrix} e^{-0.26t} + C_2 \begin{bmatrix} 2 \\ 0.46 \end{bmatrix} e^{-1.54t} + \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ 10

We apply the initial value

$I_1(0) = 0, \quad I_2(0) = 0$

$C_1 = \dfrac{\cancel{-1025 - 2750\sqrt{0.41}}}{41}$ (or $\cancel{-67.95}$ -74.74)

$C_2 = \dfrac{\cancel{-1025 + 2750\sqrt{0.41}}}{}$ (or 19.74)

$\therefore \underset{\sim}{I} = \cancel{-67.95}\ -74.74 \begin{bmatrix} 2 \\ 1.74 \end{bmatrix} e^{-0.26t} + \cancel{19.95}\ +19.74 \begin{bmatrix} 2 \\ 0.46 \end{bmatrix} e^{-1.54t} + \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ 12

5. Determine the radius of convergence. (*5 points per each*)

1) $\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$ 2) $\sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m x^{2m}$

i) $R = \dfrac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$

$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{(2m+1)!}{(2m+3)!}\right|$

$= \frac{1}{(2m+2)(2m+3)}$

$\therefore R = \infty$ 5

No partial points

ii) $R = \dfrac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$

$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{\left(\frac{2}{3}\right)^{m+1}}{\left(\frac{2}{3}\right)^m}\right| = \frac{2}{3}$

$\therefore R = \frac{3}{2}$ 3

but, in this case the series is $\square x^0 + \square x^2 + \square x^4 + \cdots$

so, the radius of convergence of x is $\sqrt{\frac{3}{2}}$ 5

6. Solve $(x^2 - x)y'' - xy' + y = 0$ using the Frobenius method. (*20 points*)

This is example 3 in page 184.
See page 184 for the details of the solution

1) Obtain y_1. (*10 points*)

$y = x^r \sum_{m=0}^{\infty} a_m x^m$

then you get $r_1 = 1,\ r_2 = 0$ 3

$a_{s+1} = \frac{s^2}{(s+1)(s+2)} a_s$ 8

let $a_0 = 1$ then $y_1 = x$ 10

2) Obtain y_2 by applying the method of reduction of order. (*10 points*)

$y_2 = y_1 \int (y_1^{-2} e^{-\int p\,dx})\,dx$ 2

where the original solution is $y'' - \frac{x}{x^2-x} y' + \frac{y}{x^2-x} = 0$

$p = -\frac{x}{x^2-x}$

$y_1 = x$

insert these to the above formular 4

then $y_2 = x \ln x + 1$ 10

$\therefore y = c_1 x + c_2 (x \ln x + 1)$

7. For the following linear system, (*20 points*)

$$-2w + x - y = 1$$
$$w - 2x + z = -5$$
$$w - 2y + z = -7$$
$$x + y - 2z = 7$$

1) Make an augmented matrix and determine the rank of the matrix. (*5 points*)

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 1 \\ -2 & 0 & 1 & 1 & -5 \\ 0 & -2 & 1 & 1 & -7 \\ 1 & 1 & -2 & 0 & 7 \end{array}\right]$$

2

$\Rightarrow$ interchange row 3 & row 3 $\Rightarrow$ make an upper triangular matrix by Gauss eliminatin

$\Downarrow$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 1 \\ 0 & -2 & 1 & -3 & -3 \\ 0 & 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 4 & -4 \end{array}\right]$$

4

rank 4 5

2) Verify if this linear system has only one set of solutions using the Cramer's rule. (5 points)

determinant $= 1 \times -2 \times -1 \times 4 \neq 0$ 5

no partial point

2) Obtain the solution by applying the method of Gauss–Jordan elimination. (*10 points*)

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & -2 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & -2 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} \\ 0 & -2 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & \frac{1}{2} & \frac{3}{4} & \frac{1}{4} & 1 \\ 0 & 0 & 0 & 4 & -2 & -1 & 1 & 0 \end{array}\right]$$

then do Gauss-Jordan eliminatin

$\Uparrow$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{4} & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & 0 \end{array}\right]$$

inverse

7

$$\therefore \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \text{inverse} \times \begin{bmatrix} 1 \\ -5 \\ -7 \\ 7 \end{bmatrix}$$

$\therefore x = 1,\ y = 2$

$z = -2\ \ w = 1$

10